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TIME-DELAY NETWORKS
FOR AN ANALOG COMPUTER

Dunham Laboratory
Yale University
New Haven, Connecticut

TIME-DELAY NETWORKS
FOR AN ANALOG COMPUTER

W. J. Cunningham

Office of Naval Research, Nonr-433(00)
Report No. 6

Dunham Laboratory
Yale University
New Haven, Connecticut

Submitted: August 1, 1954

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Preface

This report is the sixth concerned with research accomplished in connection with Navy Contract Nonr-433(00), between Dunham Laboratory, Yale University, and the Office of Naval Research, Department of the Navy. In this report is given a discussion of the design of a class of networks that can be used for producing a known constant time delay with an electronic analog computer. A differential-difference equation of the sort considered in Report No. 5 requires such a delay network if it is to be studied simply on a computer. It was in connection with the investigation of this equation that a study of delay networks was made.

The research was carried on and the report written by the undersigned.

W. J. Cunningham

New Haven, July 1954

Table of Contents

List of Figures and Tables

Abstract

In making use of an analog computer to study differential-difference equations in which a constant time delay occurs, it is necessary to have some device to introduce a known constant delay into the system. Usually it is convenient to do this with a network synthesized from the elements of the computer. A perfect delay network is an all-pass network in which phase shift varies linearly with frequency. One class of networks that approaches this ideal is studied by considering the roots of their transfer functions. Networks having one, two, three, and four pairs of roots are investigated in detail, and numerical data about them are presented in tabular form. These tables allow selection of a particular network appropriate to the problem being analyzed.

I. Need for Delay Networks

Certain physical systems have within them some process that introduces a time delay of constant magnitude, regardless of the operating conditions. The differential equations describing such a system contain terms in which the constant time delay occurs. If these equations are to be studied by means of an electronic analog computer, it is necessary to have available some device that will provide the delay. This device should produce a constant delay of controllable magnitude, and must give no effect other than that of introducing the delay.

There are several ways by which a constant time delay can be obtained. For many purposes, however, it is most convenient to synthesize from the elements of the analog computer itself a network which will produce the delay. This network is required to transmit the electrical signal representing the solution for the equation being studied. Within the frequency band containing the signal, the network should introduce constant known values of both time delay and attenuation. Such a network is described as an all-pass network with its phase shift varying linearly with the frequency of an applied sinusoidal signal. The linear relation between phase shift and frequency should be maintained over the necessary frequency band.

Time-delay networks of varying complexity can be designed. In most applications it is desirable to use as few elements as possible in the network, and to obtain as large a value as possible for the product of the time delay τ and bandwidth ω_m. One design for a time-delay network[1] has appeared recently in the literature. In the

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1. C. D. Morrill, Trans. IRE, EC-3, 45, (June 1954)

following discussion, a somewhat different approach to the problem is used, and several networks are analyzed. The number of elements and the circuitry needed to synthesize them are shown, and the value of product $\omega_m\tau$ is obtained. The results of the analysis are useful in selecting a delay network appropriate for a given application.

II. Analysis of Networks

II.1 All-pass networks

The networks considered here are all of one general type. They are all-pass networks, in which the attenuation of a sinusoidal signal is independent of frequency. Furthermore, the networks are adjusted so that this attenuation is zero, or that the magnitude of the signal is unchanged by the network. The curve relating phase shift to frequency is essentially linear from zero frequency to some maximum angular frequency ω_m. For frequencies higher than this maximum, the phase curve departs rapidly from linearity. The time delay τ is the negative of the slope of the phase curve

$$\tau = -d\theta/d\omega \tag{1}$$

where θ is the total phase shift. The delay is constant so long as the phase curve is linear.

The transfer function H for such a network is

$$H = E_2(p)/E_1(p) \tag{2}$$

where E_2 and E_1 are the complex signals at the output and input terminals, respectively, and p is the complex frequency variable. In order to fit the requirements listed previously, the magnitude of H should be unity and its angle should vary linearly with real frequency. The variation of H with p can be studied most easily through a consideration of the roots of the polynomials in p that

form the numerator and denominator of the fraction representing H. It has been shown[2] that these roots must fit a specified pattern. They must occur either as real quantities or as complex conjugate pairs. Roots of the denominator, or poles, must occur in the left half of the complex-frequency plane, and roots of the numerator, or zeros, must occur in the right half-plane. Each pole must have a matching zero with its real part of opposite algebraic sign. A typical situation is shown in Fig. 1, where six roots appear. The design problem for the delay network is essentially that of determining the optimum location for these roots.

II.2 Example with six roots

The analysis of the network with the six roots of Fig. 1 is used as an example. The transfer function for this network is

$$H = \frac{[p - (a + jb)][p - (a - jb)][p - c]}{[p + (a - jb)][p + (a + jb)][p + c]} . \tag{3}$$

If a real driving frequency is used, $p = j\omega$, and the angle of H is

$$\theta = \tan^{-1}(b - \omega)/a + \tan^{-1}(-b - \omega)/a + \tan^{-1}(-\omega/c)$$
$$- \tan^{-1}(-b + \omega)/a - \tan^{-1}(b + \omega)/a - \tan^{-1}(\omega/c) . \tag{4}$$

It is convenient to normalize some of the quantities by using the definitions

$$x \equiv b/a, \qquad y \equiv c/a, \qquad \Omega \equiv \omega/a \tag{5}$$

so that Eq. (4) becomes

$$\theta = 2\left[\tan^{-1}(x - \Omega) - \tan^{-1}(x + \Omega) - \tan^{-1}(\Omega/y)\right]. \tag{6}$$

The procedure now is to take successive derivatives of θ with respect to the frequency variable, Ω. The first derivative is

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2. H. W. Bode, Network Analysis and Feedback Amplifier Design, (D. Van Nostrand, New York, 1945), p. 239

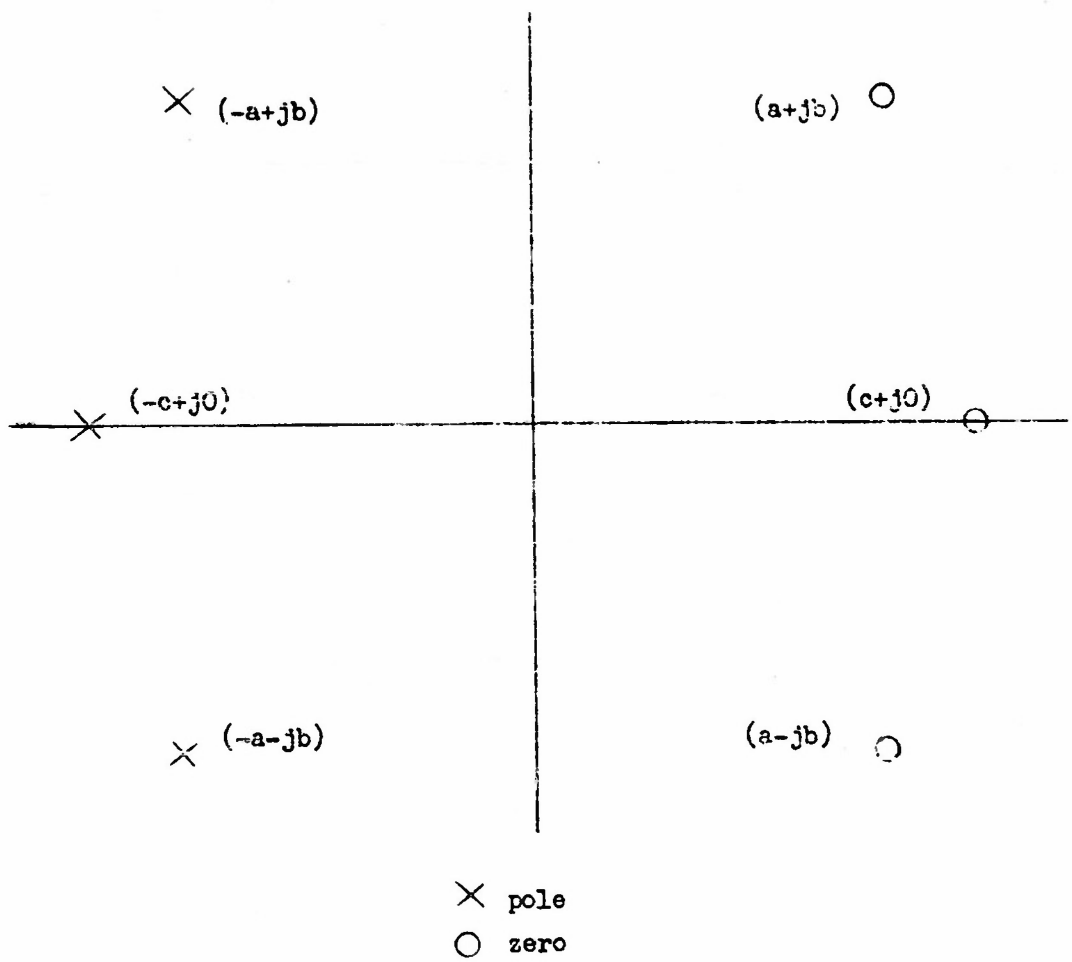


Fig. 1 Location of poles and zeros for 6-root network

proportional to the time delay τ. At zero frequency the delay is defined as τ_o, the initial time delay. Derivatives of even order all vanish at zero frequency because the phase curve is an odd function of frequency. The parameters x and y are chosen so as to make the derivatives of third and fifth order vanish at zero frequency. This insures that the phase curve will be as nearly linear as possible near zero frequency, and should provide a valid criterion for the design of the network.

The successive derivatives of Eq. (6) are as follows.

$$\frac{d\theta}{d\Omega} = -2\left[\frac{1}{1 + (x - \Omega)^2} + \frac{1}{1 + (x + \Omega)^2} + \frac{y}{y^2 + \Omega^2}\right] \tag{7}$$

$$\frac{d^2\theta}{d\Omega^2} = -4\left[\frac{(x - \Omega)}{\left[1 + (x - \Omega)^2\right]^2} - \frac{(x + \Omega)}{\left[1 + (x + \Omega)^2\right]^2} - \frac{y\Omega}{\left[y^2 + \Omega^2\right]^2}\right] \tag{8}$$

$$\frac{d^3\theta}{d\Omega^3} = 4\left[\frac{1 - 3(x - \Omega)^2}{\left[1 + (x - \Omega)^2\right]^3} + \frac{1 - 3(x + \Omega)^2}{\left[1 + (x + \Omega)^2\right]^3} + \frac{y(y^2 - 3\Omega^2)}{\left[y^2 + \Omega^2\right]^3}\right] \tag{9}$$

$$\frac{d^4\theta}{d\Omega^4} = 48\left[\frac{(x - \Omega)\left[1 - (x - \Omega)^2\right]}{\left[1 + (x - \Omega)^2\right]^4} - \frac{(x + \Omega)\left[1 - (x + \Omega)^2\right]}{\left[1 + (x + \Omega)^2\right]^4} - \frac{y\Omega(y^2 - \Omega^2)}{\left[y^2 + \Omega^2\right]^4}\right] \tag{10}$$

$$\frac{d^5\theta}{d\Omega^5} = -48\left[\frac{1 - 10(x - \Omega)^2 + 5(x - \Omega)^4}{\left[1 + (x - \Omega)^2\right]^5} + \frac{1 - 10(x + \Omega)^2 + 5(x + \Omega)^4}{\left[1 + (x + \Omega)^2\right]^5} + \frac{y(y^4 - 10y^2\Omega^2 + 5\Omega^4)}{\left[y^2 + \Omega^2\right]^5}\right] \tag{11}$$

It is evident from Eqs. (8) and (10) that at $\Omega = 0$ both $D^2\theta = 0$ and $D^4\theta = 0$, where $D^n\theta$ is defined as $d^n\theta/d\Omega^n$ evaluated at $\Omega = 0$. The odd-order derivatives evaluated at $\Omega = 0$ become

$$D\theta = -2\left[\frac{2}{1 + x^2} + \frac{1}{y}\right] \tag{12}$$

$$D^3\theta = 4\left[\frac{2(1 - 3x^2)}{\left[1 + x^2\right]^3} + \frac{1}{y^3}\right] \tag{13}$$

$$D^5\theta = -48\left[\frac{2(1 - 10x^2 + 5x^4)}{\left[1 + x^2\right]^5} + \frac{1}{y^5}\right] \tag{14}$$

The requirement that $D^3\theta = 0$ and $D^5\theta = 0$ gives

$$y^3 = -(1 + x^2)^3/2(1 - 3x^2) \tag{15}$$

and

$$y^5 = -(1 + x^2)^5/2(1 - 10x^2 + 5x^4). \tag{16}$$

If y is eliminated from these two equations, the result is

$$(1 - 10x^2 + 5x^4)^3 = 4(1 - 3x^2)^5. \tag{17}$$

The value of x satisfying this equation is $x = 0.953$, and when this value is used in either of Eqs. (15) or (16), the corresponding value for y is $y = 1.26$. These numerical values serve to give the optimum locations for the roots of the network.

The initial time delay is, from Eq. (1),

$$\tau_o = -(1/a)D\theta. \tag{18}$$

With the numerical values of parameters x and y found above, the time delay is found from Eq. (12) as

$$\tau_o = 3.68/a. \tag{19}$$

In order to determine the frequency band over which the delay is essentially constant, it is necessary to plot a curve of θ as a function of Ω. Equation (6), with the numerical values of x and y inserted, is

$$\theta = 2\left[\tan^{-1}(.953 - \Omega) - \tan^{-1}(.953 + \Omega) - \tan^{-1}(\Omega/1.26)\right]. \tag{20}$$

This equation is plotted in Fig. 2. Also plotted there is the straight line having the slope found from Eq. (12). The curve of Eq. (20)

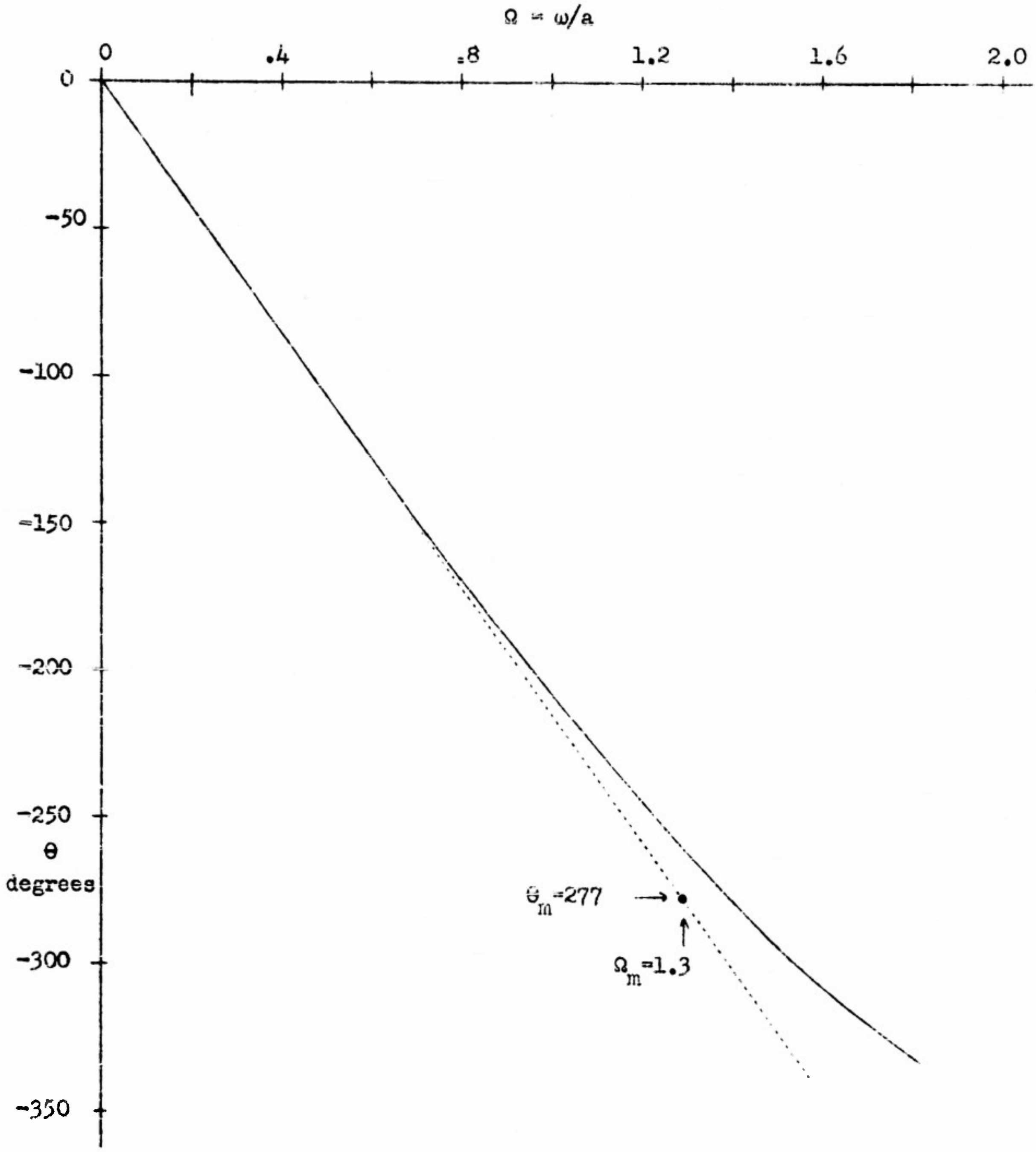


Fig. 2 Phase shift of 6-root network

departs from the straight line as Ω becomes larger. Purely arbitrarily, a difference of five percent might be chosen as an allowable departure. This departure occurs for $\Omega = 1.3$. Thus, the upper frequency limit, defined in this way, is $\omega_m = 1.3\,a$, and the product $\omega_m \tau_o$ is a sort of figure of merit, $\omega_m \tau_o = 4.8$. An alternate form for the figure of merit is the value of the phase shift when the departure from linearity is the arbitrary five percent. Here, this is $\theta_m = 277$ degrees, which is merely $\omega_m \tau_o$ expressed in degrees. It is the value θ would have at ω_m if the phase curve were truly linear. The actual value of θ is five percent less than this θ_m.

III.3 Other networks

All-pass time-delay networks, based on this type of design, obviously can be created with any integral number of pairs of roots. The example just studied contains three pairs. Where the number of pairs is odd, one pair must be located on the real axis of the p-plane. Where the number of pairs is even, all of them will be located off the axis. The complexity of the physical system needed to produce the required transfer function will increase fairly rapidly as the number of roots is increased.

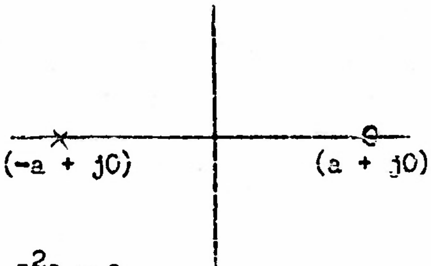
In Tables I-IV are given the results of the analysis of networks of this sort having one, two, three, and four pairs of roots. By the time four pairs of roots are used, the numerical work needed to find the optimum parameters becomes quite tedious. The performance of this 8-root network is adequate to satisfy many applications, and more complicated networks are not considered here.

Phase curves for each of the networks are plotted in Fig. 3.

Table I. 2-root Network

Pairs of roots: one

Location:



Design criteria: $D^2\theta = 0$

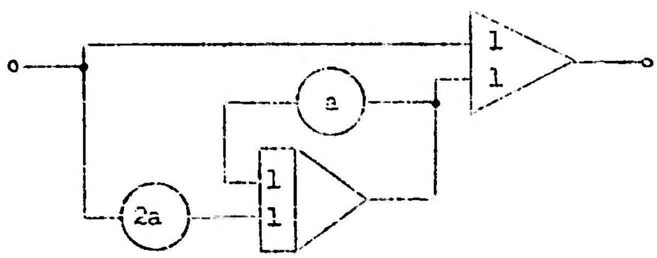
Circuit parameters: only a is specified

Initial delay: $\tau_o = 2/a$

Maximum frequency: $\omega_m = 0.4a$

Figure of merit: $\omega_m \tau_o = 0.8$, $\theta_m = 46$ degrees

Circuit:

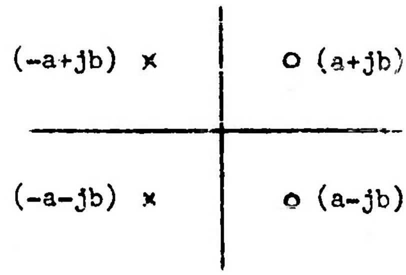


Transfer function: $H = -\frac{(p - a)}{(p + a)}$

Table II. 4-root Network

Pairs of roots: two

Location:



Design criteria: $D^2\theta = D^3\theta = D^4\theta = 0$

for $D^3\theta = 0$: $(1 - 3x^2) = 0$

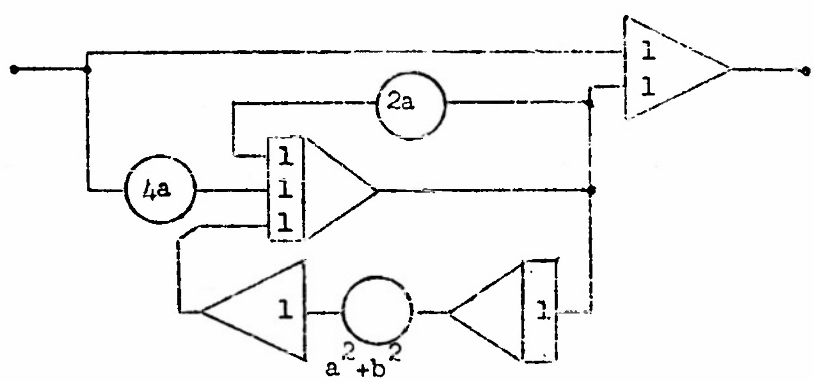
Circuit parameters: $x = b/a = 0.577$

Initial delay: $\tau_o = 3/a$

Maximum frequency: $\omega_m = 0.9a$

Figure of merit: $\omega_m \tau_o = 2.7$, $\theta_m = 155$ degrees

Circuit:

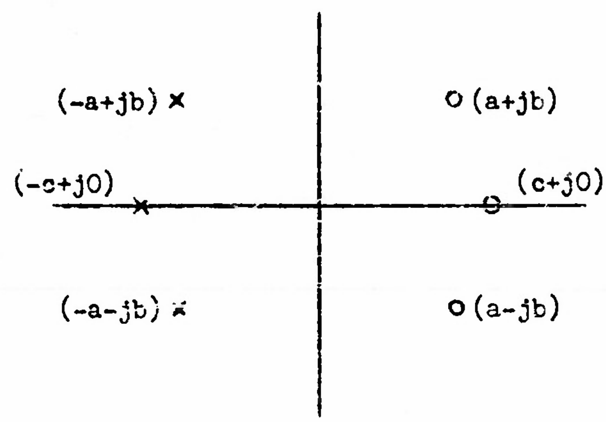


Transfer function: $H = -\dfrac{[p^2 - 2ap + (a^2 + b^2)]}{[p^2 + 2ap + (a^2 + b^2)]}$

Table III. 6-root Network

Pairs of roots: three

Location:



Design criteria: $D^2\theta = D^3\theta = D^4\theta = D^5\theta = D^6\theta = 0$

for $D^3\theta = 0$: $\dfrac{2(1 - 3x^2)}{(1 + x^2)^3} + \dfrac{1}{y^3} = 0$

for $D^5\theta = 0$: $\dfrac{2(1 - 10x^2 + 5x^4)}{(1 + x^2)^5} + \dfrac{1}{y^5} = 0$

Circuit parameters: $x = b/a = 0.953$

$y = c/a = 1.26$

Initial delay: $\tau_o = 3.68/a$

Maximum frequency: $\omega_m = 1.3\,a$

Figure of merit: $\omega_m\tau_o = 4.8$, $\theta_m = 277$ degrees

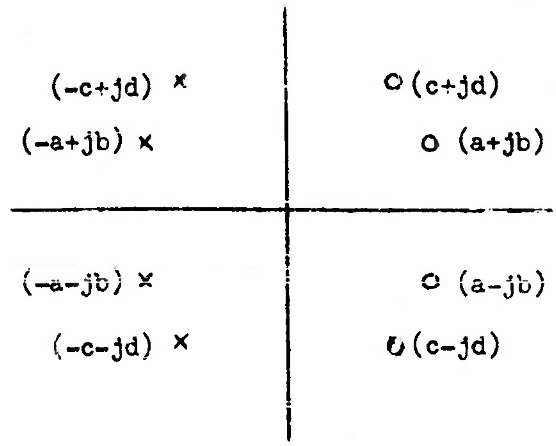
Circuit: The circuit of Table I, designed with parameter c, followed by the circuit of Table II, designed with parameters a and b.

Transfer function: $H = \dfrac{[p^2 - 2ap + (a^2 + b^2)][p - c]}{[p^2 + 2ap + (a^2 + b^2)][p + c]}$

Table IV. 8-root Network

Pairs of roots: four

Location:



Design criteria: $D^2\theta = D^3\theta = D^4\theta = D^5\theta = D^6\theta = D^7\theta = D^8\theta = 0.$

for $D^3\theta = 0$: $$\frac{1 - 3x^2}{(1 + x^2)^3} + \frac{1 - 3z^2}{y^3(1 + z^2)^3} = 0$$

for $D^5\theta = 0$: $$\frac{1 - 10x^2 + 5x^4}{(1 + x^2)^5} + \frac{1 - 10z^2 + 5z^4}{y^5(1 + z^2)^5} = 0$$

for $D^7\theta = 0$: $$\frac{1 - 21x^2 + 35x^4 - 7x^6}{(1 + x^2)^7} + \frac{1 - 21z^2 + 35z^4 - 7z^6}{y^7(1 + z^2)^7} = 0$$

Circuit parameters: $x = b/a = 0.34$

$y = c/a = 0.71$

$z = d/c = 1.5$

Initial delay: $\tau_o = 5.26/a$

Maximum frequency: $\omega_m = 1.55a$

Figure of merit: $\omega_m \tau_o = 8.2$, $\theta_m = 470$ degrees

Circuit: The circuit of Table II, designed with parameters a and b, followed by a second circuit of Table II, designed with parameters c and d.

Transfer function: $$H = \frac{[p^2 - 2ap + (a^2 + b^2)][p^2 - 2cp + (c^2 + d^2)]}{[p^2 + 2ap + (a^2 + b^2)][p^2 + 2cp + (c^2 + d^2)]}$$

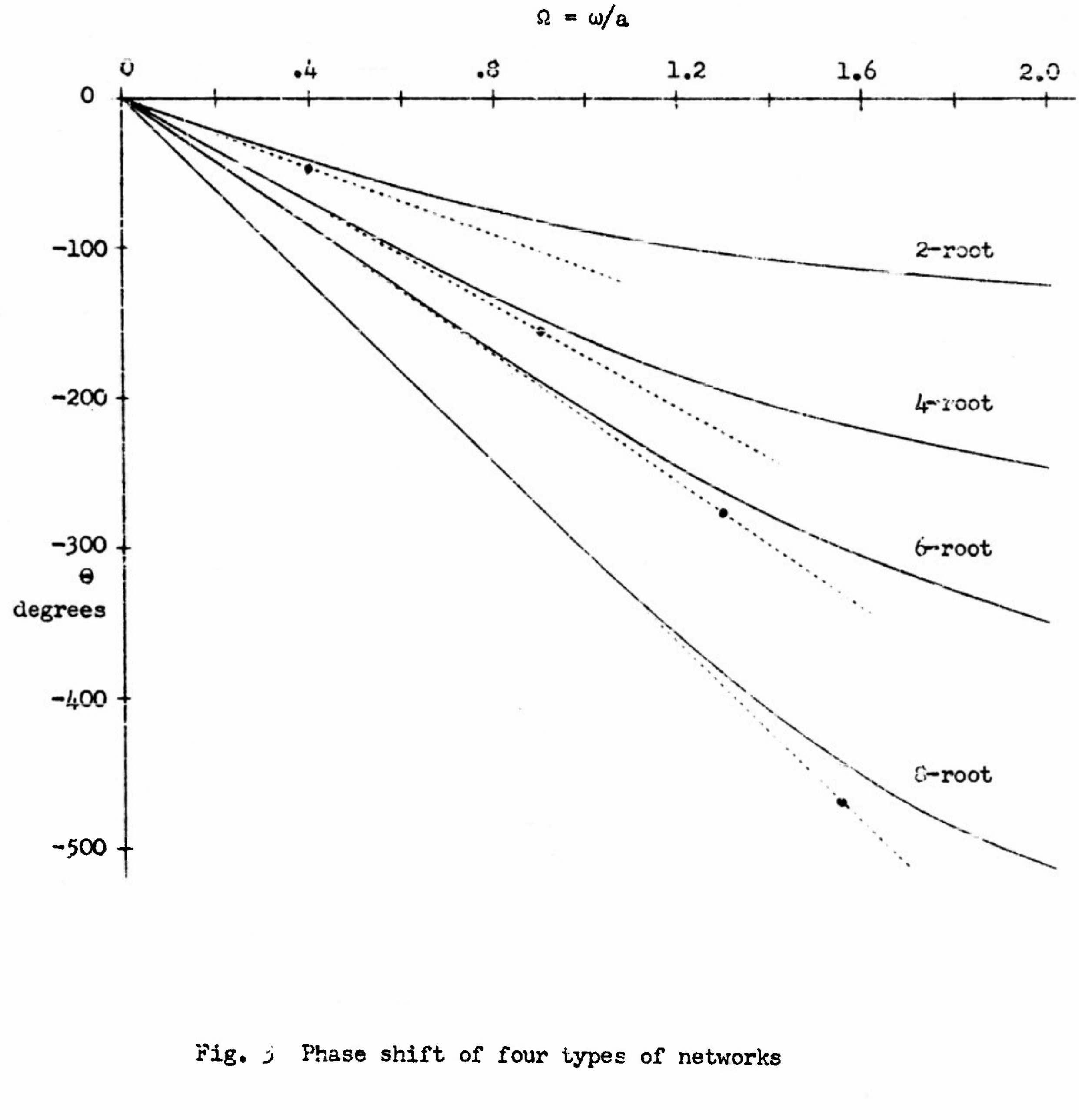


Fig. 5 Phase shift of four types of networks

III. Synthesis of Networks

The elements of an electronic analog computer must now be used to synthesize a network having its roots determined by the foregoing analysis. Since the computing amplifiers are unilateral circuit elements, the synthesis is not difficult to carry out. One circuit is sufficient to give a single pair of roots on the real axis. A second circuit will give two pairs of roots symmetrically located off the axes. These two circuits can then be combined to give any arrangement of roots that may be desired.

III.1 One pair of roots

The transfer function for the time-delay network with a single pair of roots on the real axis is

$$H = \frac{(p - a)}{(p + a)} \tag{21}$$

which can be written as

$$H = 1 - 2a/(p + a) . \tag{22}$$

The term, $-1/(p + a)$, can be obtained by applying resistive feedback around an integrator. If this term is adjusted in magnitude by the factor $2a$ and added to unity, the result is the desired transfer function. The corresponding circuit is shown in Table I. The circuit yields a transfer function which is the same as Eq. (21), but with a negative algebraic sign.

III.2 Two pairs of roots

The transfer function for the network with two pairs of roots is

$$H = \frac{(p - a - jb)(p - a + jb)}{(p + a + jb)(p + a - jb)} = \frac{p^2 - 2ap + (a^2 + b^2)}{p^2 + 2ap + (a^2 + b^2)} \tag{23}$$

which can be written as

$$H = 1 - 4ap/\left[p^2 + 2ap + (a^2 + b^2)\right]. \qquad (24)$$

The term, $-p/\left[p^2 + 2ap + (a^2 + b^2)\right]$, can be obtained by applying both resistive feedback and feedback with one integration around an integrator. If this term is adjusted by the factor $4a$ and added to unity, the result is the desired transfer function. The corresponding circuit is shown in Table II. This circuit yields a transfer function which is the same as Eq. (23), but with a negative algebraic sign.

III.3 Several pairs of roots

When several pairs of roots are necessary, it seems simplest to combine the circuits of Tables I and II in appropriate fashion. Thus, the circuits of Tables I and II are combined in Table III; two circuits of Table II are combined in Table IV.

It is possible, of course, to synthesize other arrangements of circuit elements that will give the transfer functions with several pairs of roots. The alternate configurations that have been explored require at least as many circuit elements as the combinations suggested in Tables III and IV, and appear to be more difficult to adjust. Where a complicated circuit is obtained as the sum of several simpler circuits, each circuit can be tested alone to make sure it is operating correctly. Alternate forms of the complicated circuit may not be so easily broken down for test purposes.

IV. Practical Considerations

The primary consideration in choosing the appropriate delay network for a particular application is the total phase shift that must be obtained with a curve essentially linear. This is the form of the

figure of merit given as θ_m in the tables. A network with a larger number of roots gives a larger value for θ_m. Thus, if a large value of θ_m is needed for a certain problem, a relatively complicated delay network is required.

Two identical networks might be used in tandem to give double the delay of either alone. In Fig. 4 is shown the phase curve for a pair of identical 4-root networks used in this way. Also shown is the curve for a single 8-root network. The initial delay for the pair of 4-root networks is twice that for one alone, and is larger than that for the 8-root network. However, the departure from linearity is greater than for the single 8-root network. The figure of merit for the pair of 4-root networks is twice 2.7, or 5.4, while that for the single 8-root network is 8.2. This example indicates that it is advantageous to use a single network of sufficient complexity to give the necessary total phase shift.

In setting up the components to form a delay circuit, it is sometimes helpful to test isolated portions of the complete system. The circuit giving four roots can be profitably broken down in this way. If the transmission path straight from the input terminal of the network to one input terminal of the summing amplifier is removed, the transfer function of the remaining elements is

$$G = j4a\omega/\left[(a^2 + b^2 - \omega^2) + j2a\omega\right]. \tag{25}$$

It is not difficult to show that its magnitude is $|G| = 2$ for the driving frequency $\omega^2 = (a^2 + b^2)$. Similarly its magnitude is $|G| = 2^{1/2}$ for the driving frequencies $\omega^2 = (3a^2 + b^2) \pm 2a(2a^2 + b^2)^{1/2}$. This portion of the network can easily be checked to make sure that $|G|$ has the proper value at the frequencies given.

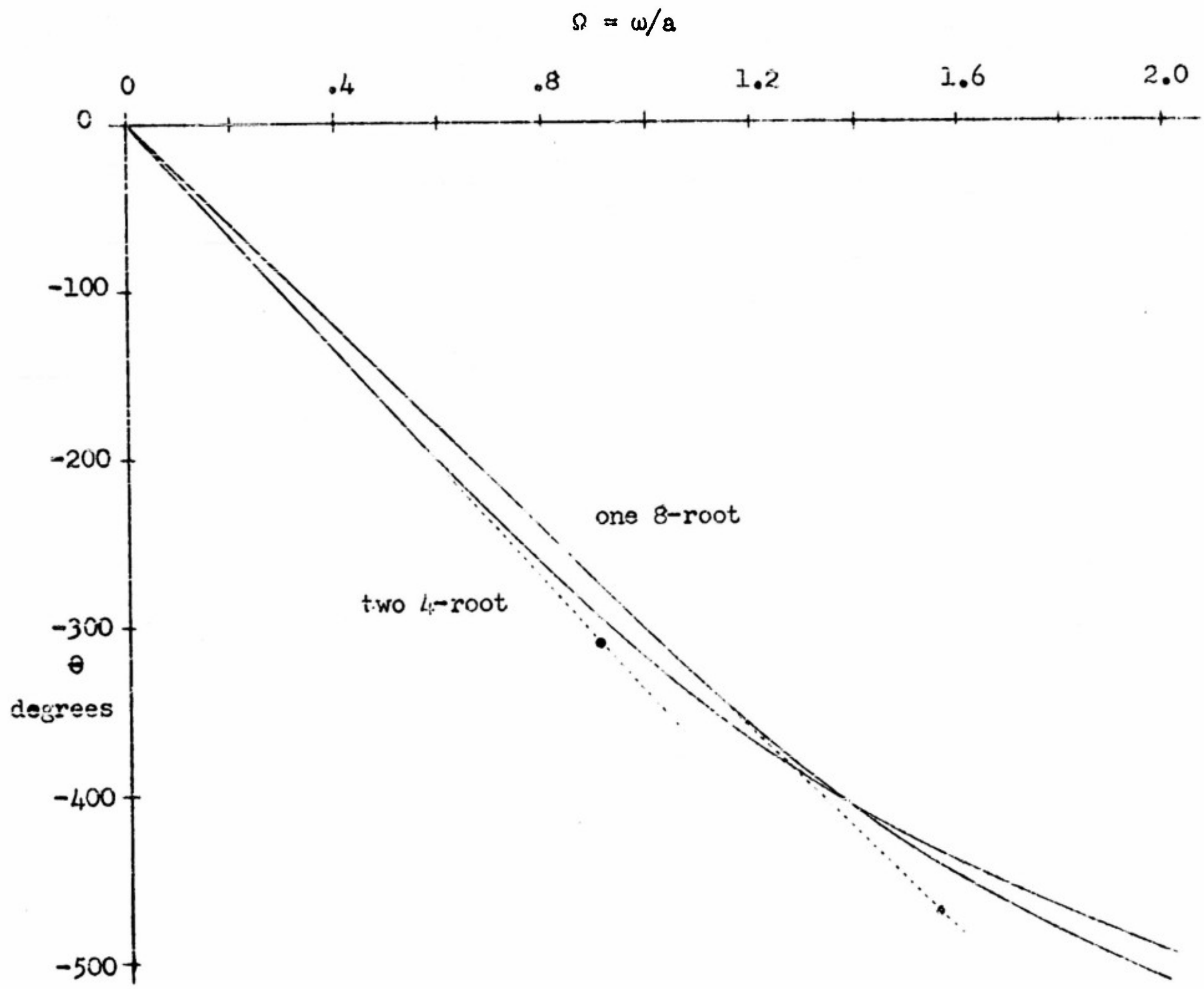


Fig. 4 Phase shift of two 4-root networks and of one 8-root network

The magnitude of the transfer function for any of the complete delay networks is, of course, unity at any driving frequency, and this fact should be checked.

V. Example of Application

A nonlinear differential-difference equation requiring a time delay is

$$dX(t)/dt = A\,X(t) - B\,X(t)\,X(t-\tau) \qquad (26)$$

where X is the dependent variable, t is time, the independent variable, A and B are positive real constants, and τ is a constant time delay. This equation has been studied[3] on an analog computer, using a step-by-step process of computation. For certain values of product $A\tau$, the solution for the equation is a periodic oscillation with a waveform that is far from sinusoidal. The period of the oscillation is at least four, or more, times the delay time τ. Thus, a delay of somewhat less than 90 degrees is needed at the fundamental frequency. However, many harmonic components are present in the nonsinusoidal oscillation. The phase shift for a harmonic with constant time delay is the product of the order of the harmonic and the phase shift of the fundamental. Since it appears that harmonics of at least the fifth order are important, a total phase shift of at least 400 degrees is required. The use of an 8-root delay circuit is indicated.

A convenient value for the time delay with the computer used here is $\tau_o = 2$ seconds. Then, using the design equations of Table IV, the necessary parameters are

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3. W. J. Cunningham, A Nonlinear Differential-Difference Equation of Growth, Contract Nonr-433(00), Yale University, May 1954. Upper-case symbols are used in writing Eq. (26) to avoid confusion with symbols used previously.

$$a = 5.28/\tau_o = 2.64 \text{ sec}^{-1}$$
$$b = xa = .34 \times 2.64 = 0.895 \text{ sec}^{-1}$$
$$c = ya = .71 \times 2.64 = 1.87 \text{ sec}^{-1}$$
$$d = zc = 1.5 \times 1.87 = 2.81 \text{ sec}^{-1}$$
$$a^2 + b^2 = 7.72 \text{ sec}^{-2}$$
$$c^2 + d^2 = 11.41 \text{ sec}^{-2}$$

The configuration of elements needed to synthesize the delay circuit is shown in Fig. 5. The complete setup of the computer for studying Eq. (26) is shown in Fig. 6, and incorporates the delay circuit of Fig. 5.

A family of curves obtained with this system is shown in Fig. 7. The only initial condition for these curves is that $X = \beta/10$ at $t = 0$, where $\beta = A/B$. These curves agree well with those of Fig. 19 of footnote reference 3. These latter curves were plotted with points obtained from a step-by-step integration. The curves of Fig. 7 are plotted directly and continuously by the computer. The minimum value of product $A\tau$ needed to give a periodic solution is approximately $A\tau = 1.85$. This is somewhat larger than that estimated in the reference. The solution just overshoots its final value, so oscillation begins, if $A\tau = 0.48$, which is close to the previous estimate.

In Fig. 8 is shown the solution for $X(t)$ with $A\tau = 1.8$, and also this same function as retarded by the delay network. Except for the initial transient, the delayed curve is quite accurately the original curve, merely shifted the proper amount on the time scale. If the product $A\tau$ is much larger than about $A\tau = 2$, the oscillation becomes even more nonsinusoidal. The higher harmonics then exceed the maximum frequency for the delay network, and the delayed signal begins to differ in shape from the signal with no delay.

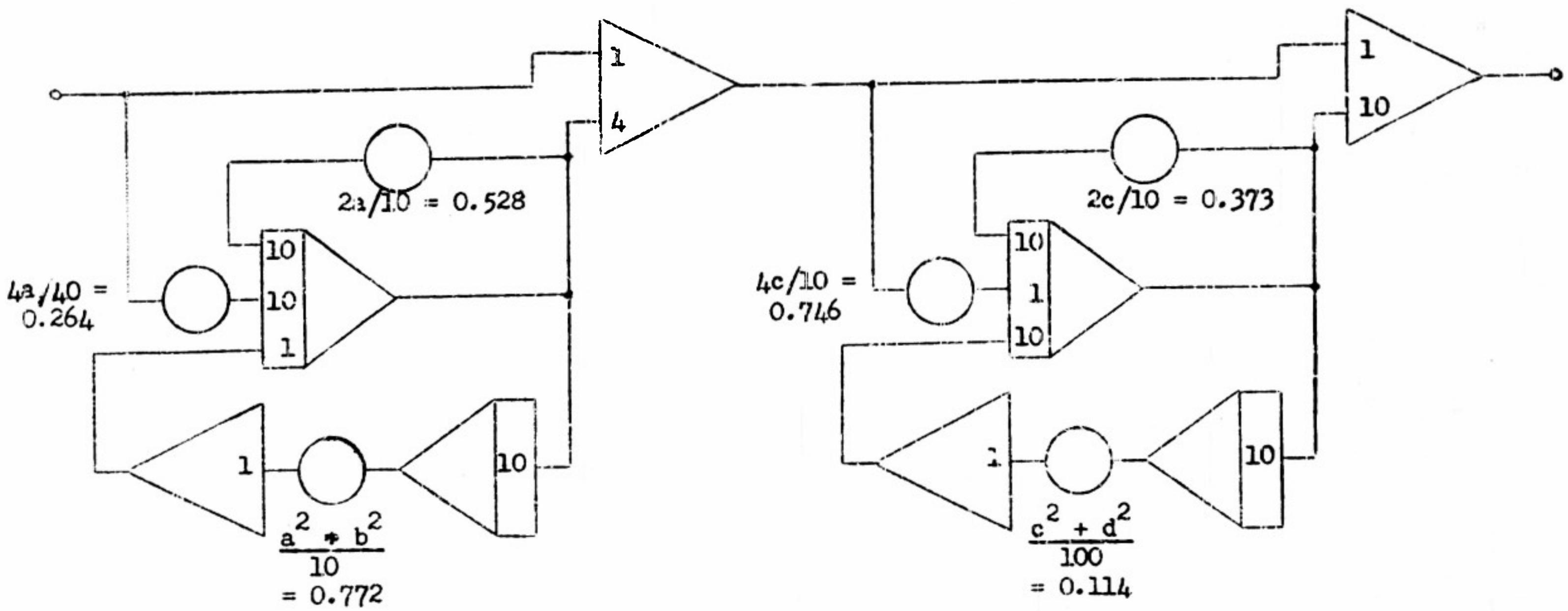

Fig. 5 Computer setup for eight roots, $\tau_o = 2$ seconds.

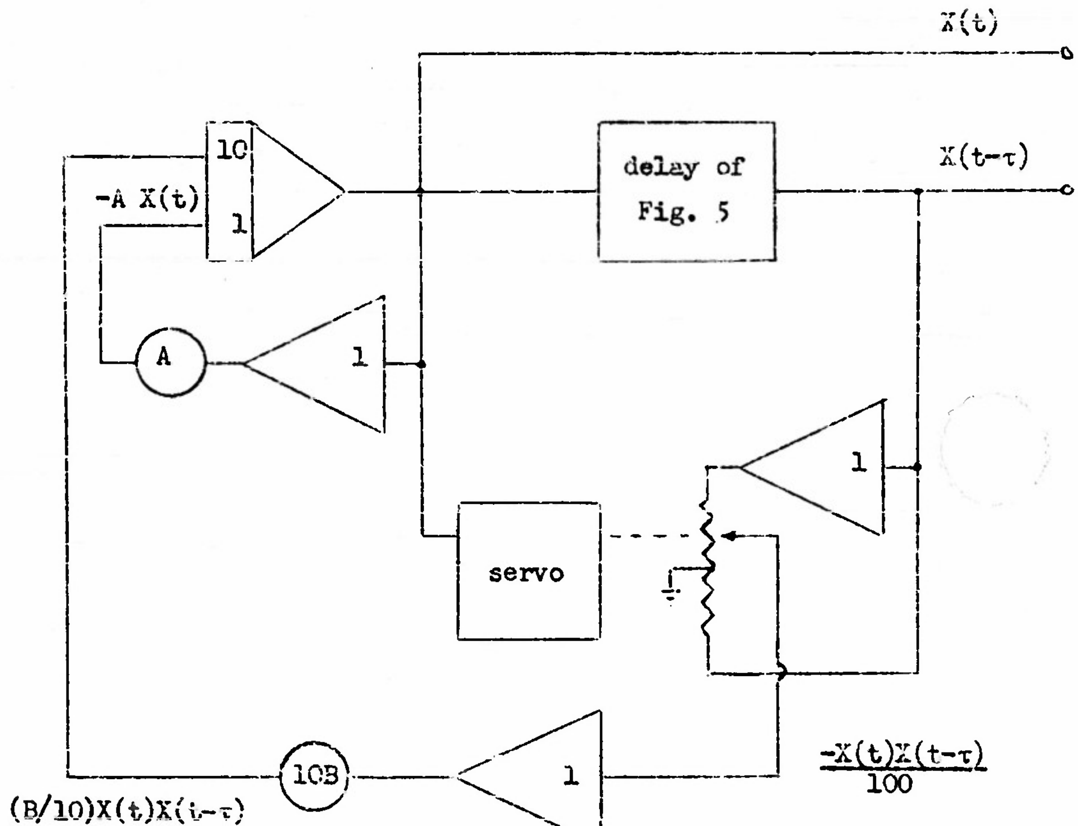

Fig. 6 Computer setup for Eq. (26)

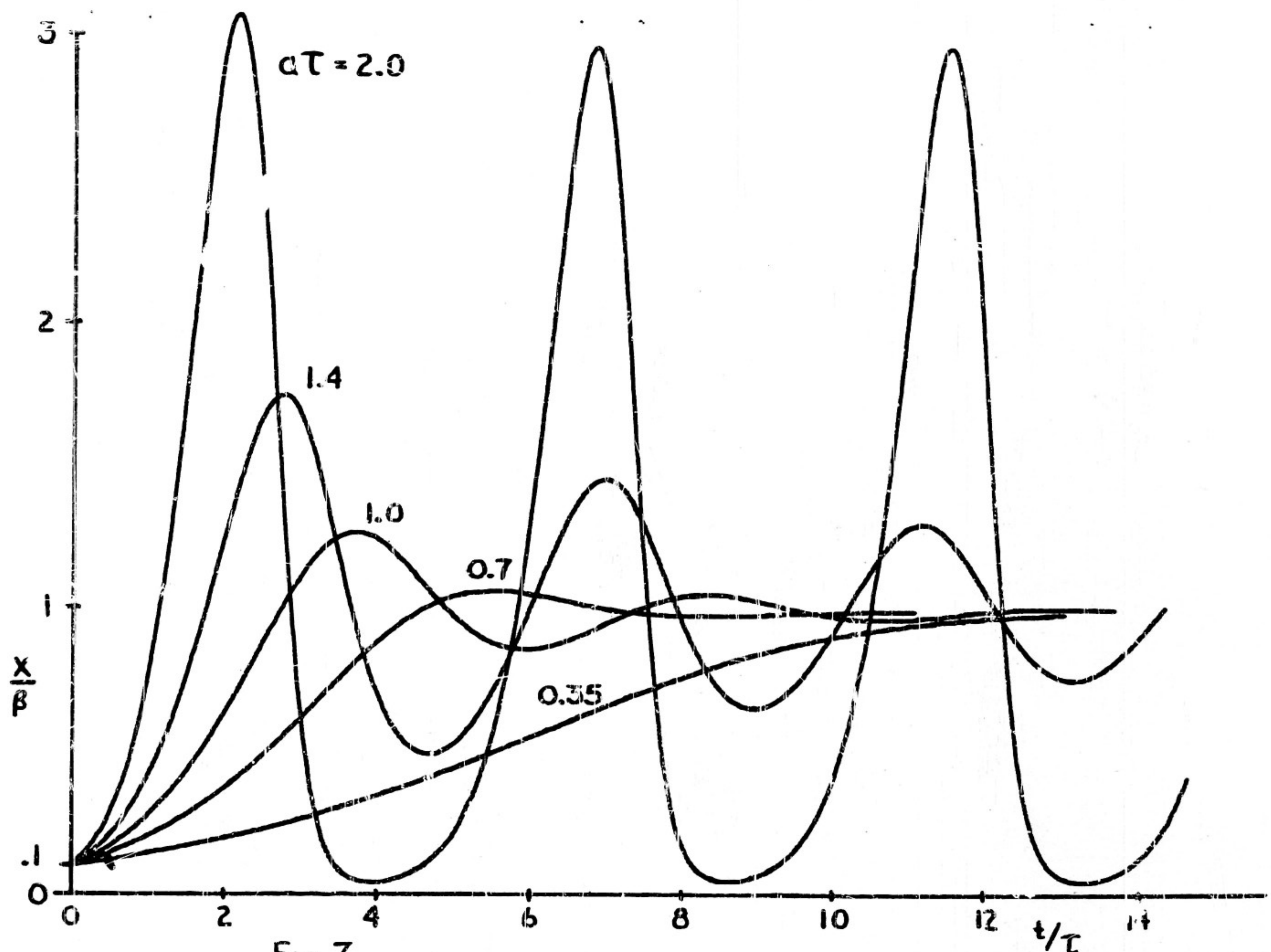

Fig. 7

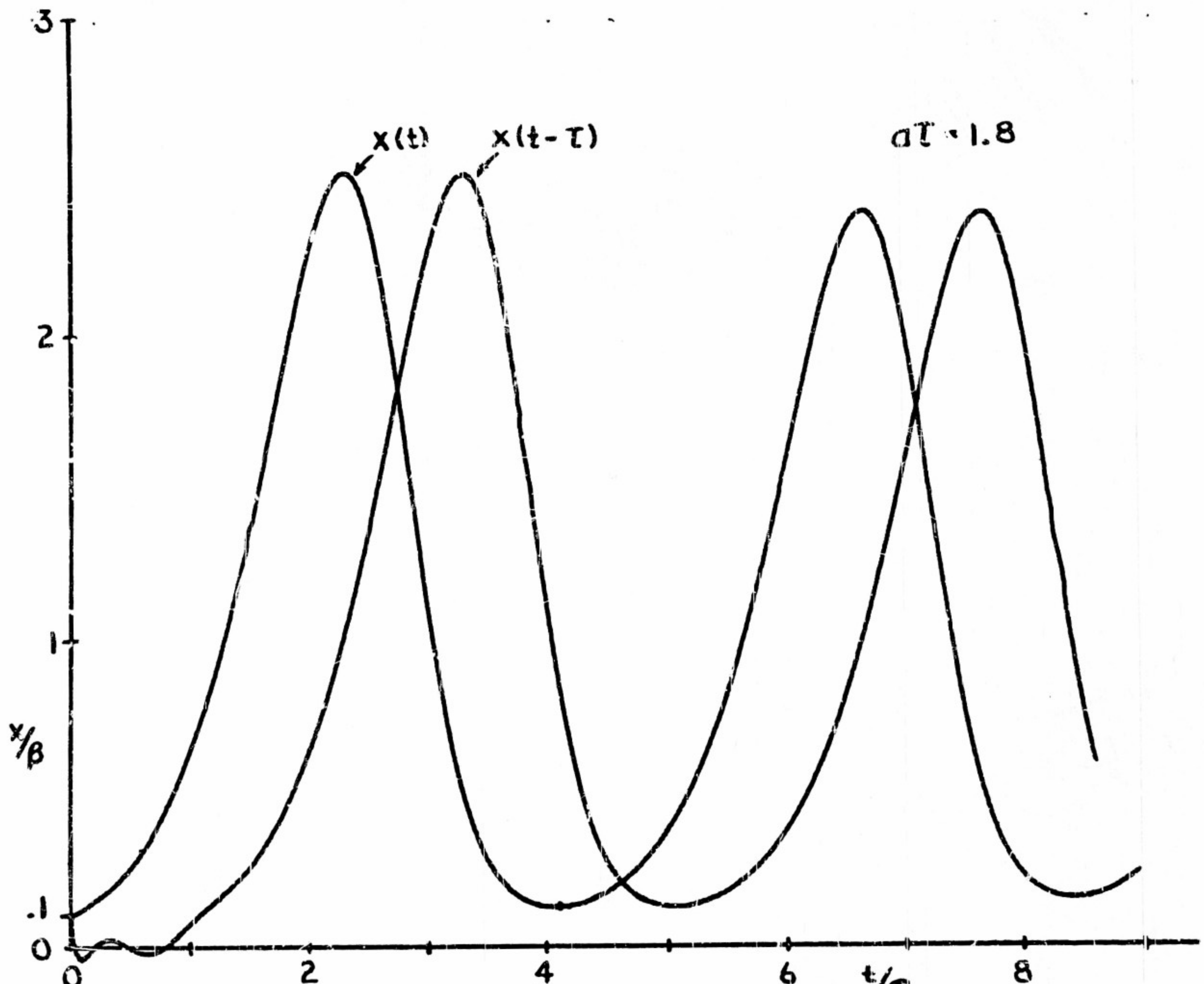

Fig. 8

The initial transient on the delayed curve of Fig. 8 comes about because no initial conditions were applied to the integrators of the delay circuit. If enough information is known about the solution of the equation for the interval $-\tau \leq t \leq 0$, presumably initial conditions needed in the delay circuit could be predicted and inserted in such a way as to avoid this initial transient. In practice, however, problems involving differential-difference equations of this sort appear to have the inherent difficulty of not furnishing sufficient information to start the solution properly. As has been observed previously, this kind of equation is equivalent to a differential equation of infinite order, so that an infinity of initial conditions are needed. With a finite delay network, such as those considered here, only a finite number of conditions can be used. Even this finite number will not be known in the general problem.

The damping effective in the delay circuit is sufficient so that if there is a transient set up within it, decay is fairly complete within the interval of one delay time, τ.

Initial conditions needed to eliminate the initial transient in the 8-root delay network can be found as follows. The network with its four integrators is shown in Fig. 9. Initial conditions applied to the integrators are identified as IC_m; these are the negatives of the signals which must exist at the output terminals of the integrators at zero time. Constant multiplying factors for certain of the computing amplifiers are identified as N_n. Signals at various points in the systems are called $e_k = e_k(t)$.

Through a consideration of the transfer functions for the parts of the network, the following equations can be written.

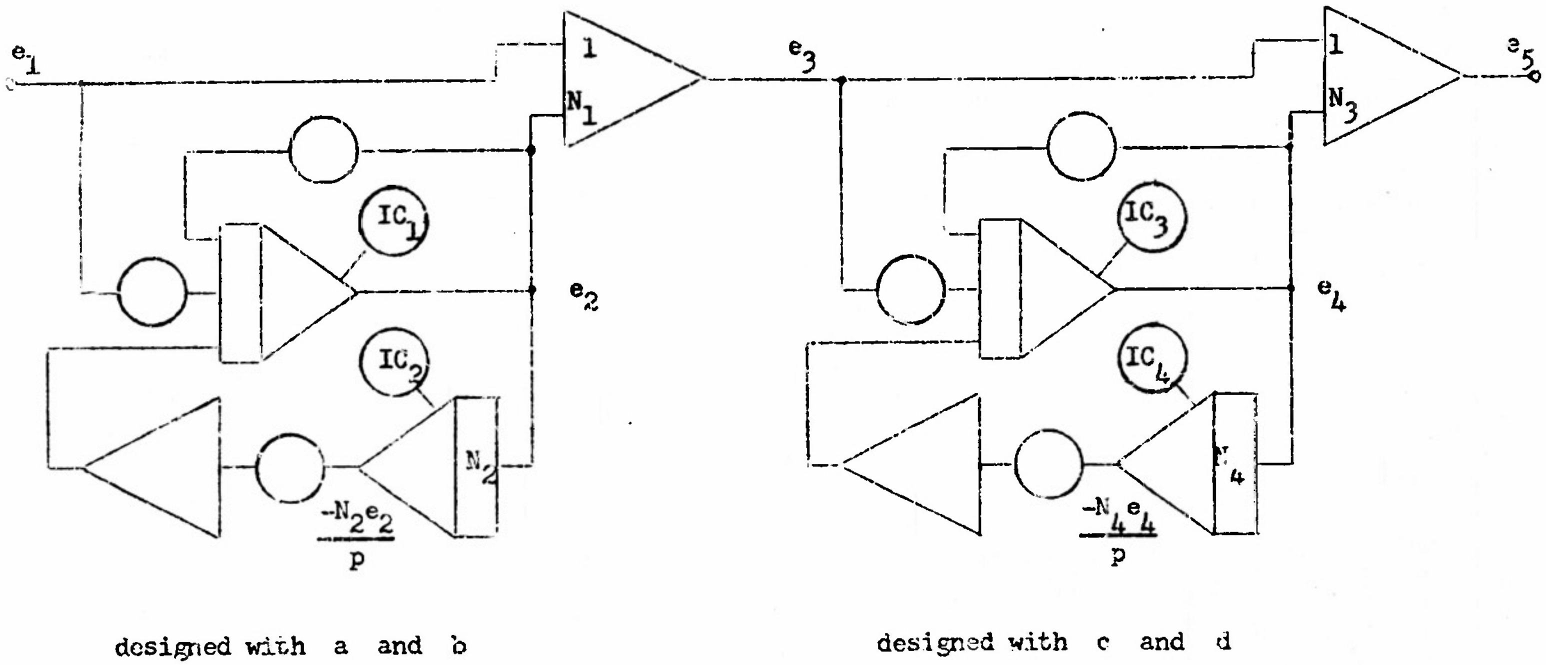

Fig. 9 Insertion of initial conditions in 8-root network

$$IC_1 = -e_2(0) = \left[\frac{(4/N_1)ap}{p^2 + 2ap + (a^2 + b^2)}\right] e_1(t)\Big|_{t=0} \tag{27}$$

$$IC_2 = \frac{N_2 e_2(0)}{p} = \left[\frac{-(4N_2/N_1)a}{p^2 + 2ap + (a^2 + b^2)}\right] e_1(t)\Big|_{t=0} \tag{28}$$

$$IC_3 = -e_4(0) = \left[\frac{(4/N_3)cp}{p^2 + 2cp + (c^2 + d^2)}\right] \left[\frac{p^2 - 2ap + (a^2 + b^2)}{p^2 + 2ap + (a^2 + b^2)}\right] e_1(t)\Big|_{t=0} \tag{29}$$

$$IC_4 = \frac{N_4 e_a(0)}{p} = \left[\frac{-(4N_4/N_3)c}{p^2 + 2cp + (c^2 + d^2)}\right] \left[\frac{p^2 - 2ap + (a^2 + b^2)}{p^2 + 2ap + (a^2 + b^2)}\right] e_1(t)\Big|_{t=0} \tag{30}$$

Enough information is known about Eq. (26) to determine these initial conditions. Near $X = 0$, approximately $X = X_o \exp(At)$, where $X = X_o$ at $t = 0$. For the curves of Fig. 8, $A = 0.9$ and $X_o = \beta/10$. Thus, for the setup of Figs. 5 and 6, and the curves of Fig. 8, numerical data needed for Eqs. (27-30) are as follows:

$a = 2.64$	$N_1 = 4$	$e_1(t) = (\beta/10)\exp(0.9t)$
$b = 0.895$	$N_2 = 10$	
$c = 1.87$	$N_3 = 10$	
$d = 2.81$	$N_4 = 10$	

When these data are inserted into the equations and the necessary operations carried out, the initial conditions for the integrators are found as

$IC_1 = 0.177\ (\beta/10)$ $\qquad$ $IC_3 = -0.0123\ (\beta/10)$

$IC_2 = -1.97\ (\beta/10)$ $\qquad$ $IC_4 = 0.137\ (\beta/10)$

When these conditions are applied, the delayed solution starts off smoothly, without an initial transient such as that of Fig. 8.

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